Infinite sets

A set A is <u>infinite</u> if it's not finite.

A is <u>countably infinite</u> if there is a bijection $f: A \to \mathbb{Z}_+$

A is <u>countable</u> if it's finite or countably infinite. Otherwise it is <u>uncountable</u>.

Examples:

1.) $\mathbb{Z}$ is countable. Define $f: \mathbb{Z} \to \mathbb{Z}_+$ by

$$f(x) = \begin{cases} 2x & \text{if } x > 0 \\ -2x+1 & \text{if } x \leq 0 \end{cases}$$

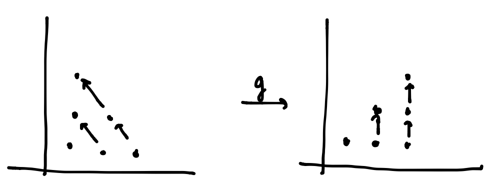
(Check that this is well-defined and a bijection)

Notice that infinite sets can have bijections w/ subsets! (Whereas finite sets can't.)

2.) $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable. Define $f: (\mathbb{Z}_+ \times \mathbb{Z}_+) \to \mathbb{Z}_+$ as follows:

First, define $g: \mathbb{Z}_+ \times \mathbb{Z}_+ \to A$, where $A = \{(x,y) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid y \leq x\}$

by $g(x,y) = (x+y-1, y)$



Then define $h: A \to \mathbb{Z}_+$ by $h(x,y) = \frac{1}{2}(x-1)x + y$.

(Then $h(1,1) = 1$, $h(2,1) = 2$, $h(2,2) = 3$, etc.)

Check that g and h are bijections. Then define $f = h \circ g$.
good exam review!!!

Infinite sets can have bijections w/ their proper subsets, but uncountable sets can't inject into uncountable sets. i.e.:

Lemma: If $C \subseteq \mathbb{Z}_+$ is infinite, then C is countable.

Pf: We define $h: \mathbb{Z}_+ \to C$ recursively:

$$h(1) = \min(C) = \text{smallest element of } C.$$
$$h(i) = \min\left(C - h(\{1, 2, \ldots, i-1\})\right).$$

This is well-defined since C is infinite (so $C - \{1, 2, \ldots, i-1\} \neq \emptyset$) and every subset of $\mathbb{Z}_+$ has a minimum element.

To show h is injective, suppose $m, n \in \mathbb{Z}_+$ s.t. $m < n$. Then $h(n) \in C - h(\{1, \ldots, m, \ldots, n-1\})$, so $h(n) \neq h(m)$.

To see that h is surjective, let $c \in C$. We know that $h(\mathbb{Z}_+)$ is infinite since h is injective, so it's not contained in $\{1, \ldots, c\}$.

Thus, $\exists$ $n \in \mathbb{Z}$ s.t. $h(n) > c$.

Let m be the smallest element of $\mathbb{Z}_+$ s.t. $h(m) \geq c$.

Then for $i < m$, $h(i) < c$.

Thus $c \notin h(\{1, \ldots, m-1\})$ so $c \in C - h(\{1, \ldots, m-1\})$.

since $h(m)$ is the smallest element of $C - h(\{1,\dots,m-1\})$, $h(m) \leq c$. Thus $h(m) = c$, so h is a bijection, as desired. $\square$

Theorem: Let B be a nonempty set. Then the following are equivalent.

1.) B is countable.

2.) $\exists$ surjective function $f: \mathbb{Z}_+ \to B$

3.) $\exists$ injective function $g: B \to \mathbb{Z}_+$.

Proof: First we show 1.) $\Rightarrow$ 2.). Assume B is countable.

If B is infinite, $\exists$ a bijection $\mathbb{Z}_+ \to B$, which is thus surjective.

If B is finite, $\exists$ a bijection $g: \{1,\dots,n\} \to B$ for some n.

Define $f: \mathbb{Z}_+ \to B$ by $f(x) = \begin{cases} g(x) & \text{if } 1 \leq x \leq n \\ g(n) & \text{otherwise} \end{cases}$

f is certainly a surjection.

2.) $\Rightarrow$ 3.): Let $f: \mathbb{Z}_+ \to B$ be a surjection. Define $g: B \to \mathbb{Z}_+$ by

$$g(b) = \min\left(f^{-1}(\{b\})\right)$$

This is well-defined since f is surjective so $f^{-1}(\{b\})$ is nonempty and every subset of $\mathbb{Z}_+$ has a minimum element.

g is injective since for $b \neq a$ in B, $f^{-1}(\{a\}) \cap f^{-1}(\{b\}) = \emptyset$.

3.) $\Rightarrow$ 1.): Let $g: B \to \mathbb{Z}_+$ be injective.

Define $g_0 : B \to g(B)$ to be the function obtained by restricting the target of g. g_0 is bijective, by construction.

$g(B) \subseteq \mathbb{Z}_+$, so it's either finite or countably infinite, so B is as well. $\square$

Def: Sets A and B have the <u>same cardinality</u> if $\exists$ a bijection $f : A \to B$.

Thus, this theorem says that if a set has any of those 3 properties, then it has the same cardinality as $\mathbb{Z}_+$

Corollary: A subset of a countable set is countable.

Pf: If $A \subseteq B$, B countable, then either B is finite, in which case A is finite, or B is countable infinite. If B is infinite, then $\exists$ a bijection $f : B \to \mathbb{Z}_+$, and $f|_A : A \to \mathbb{Z}_+$ is an injection. $\square$

Ex: Now we have an easier way to show $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable: Define $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ as

$$f(m,n) = 2^m 3^n.$$

Claim: f is injective. Suppose $2^m 3^n = 2^p 3^q$.

Then WLOG assume $m \geq p$. Then $2^{m-p} 3^n = 3^q$.

3^q is odd, so $m = p$. Thus $3^n = 3^q$, so $n = q$ as well.

Ex: Let $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Q}_+$ be defined

$$f(a,b) = \frac{a}{b}.$$

Then f is surjective and $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable.

if $g : \mathbb{Z}_+ \to \mathbb{Z}_+ \times \mathbb{Z}_+$ is a bijection, then $f \circ g : \mathbb{Z}_+ \to \mathbb{Q}_+$ is

surjective, so $\mathbb{Q}_+$ is countable. (Note that we can just replace $\mathbb{Z}_+$ in the theorem w/ any countable set.

Thm: A countable union of countable sets is countable.

Pf: Let $\{A_n\}_{n \in J}$ be a countable family of countable sets.

i.e. $J = \{1, \dots, N\}$ or $J = \mathbb{Z}_+$ and each A_n is countable.

Thus, for each n, we can choose a surjective function

$$f_n : \mathbb{Z}_+ \longrightarrow A_n.$$

We can also choose a surjection $g : \mathbb{Z}_+ \longrightarrow J$.

Define $h : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup_{n \in J} A_n$ by

$$h(i,j) = f_{g(i)}(j) \quad \text{(i.e. the } j^{th} \text{ elt of the } i^{th} \text{ set)}$$

h is surjective, since g and f are (check this!).

Thus, since $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable, $\exists$ a surjection $h_0 : \mathbb{Z}_+ \to \mathbb{Z}_+ \times \mathbb{Z}_+$, so $(h \circ h_0)$ is a surjection $\Rightarrow \bigcup_{n \in J} A_n$ is countable. $\square$

Ex: $\mathbb{Q}_+$ and $\mathbb{Q}_-$ have a natural bijection between them. Thus,

$\mathbb{Q}_-$ is countable, so $\mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\}$ is countable.

Def: Let X be a set. Define an <u>ω-tuple</u> of elements of X to be a function $\vec{x} : \mathbb{Z}_+ \to X$.

Denote $\vec{x}(i)$ as x_i, and denote the function $\vec{x}$ by

$$(x_1, x_2, \dots) \text{ or } (x_i)_{i \in \mathbb{Z}_+}.$$

$(x_1, x_2, \dots)$ is also called a sequence of elements of X.

If $\{A_i\}_{i \in \mathbb{Z}_+} = \{A_1, A_2, \dots\}$ is a collection of sets indexed by $\mathbb{Z}_+$, and $X = \bigcup_{i \in \mathbb{Z}_+} A_i$,

then the <u>cartesian product</u> of the collection, denoted

$$\prod_{i \in \mathbb{Z}_+} A_i \text{ or } A_1 \times A_2 \times \dots$$

is the set $\{(a_1, a_2, \dots) \mid a_i \in A_i \ \forall i \in \mathbb{Z}_+\}$.

If $A_i = A_j \ \forall i, j \in \mathbb{Z}_+$ then $X = A_i$ and the product is denoted X^ω.

Ex: $\mathbb{R}^m$ is the set of m-tuples of real numbers – "euclidean m-space".
$\mathbb{R}^\omega$ is the set of all ω-tuples $(x_1, x_2, \dots)$ s.t. $x_i \in \mathbb{R}$. Sometimes called "infinite-dimensional euclidean space."

Theorem: Let $X = \{0,1\}$. Then X^ω is uncountable

Proof: We'll show that given any $g : \mathbb{Z}_+ \to X^\omega$, g is not surjective.

Denote $g(n) = (x_{n1}, x_{n2}, x_{n3}, \ldots)$

Define $\vec{y} = (y_1, y_2, \ldots) \in X^{\omega}$ by

$$y_n = \begin{cases} 0 & \text{if } x_{nn} = 1 \\ 1 & \text{if } x_{nn} = 0 \end{cases}$$

Then $\forall\ n \in \mathbb{Z},\ g(n) \neq \vec{y}$, since $g(n)_n = x_{nn} \neq y_n$. Thus, $\vec{y} \notin f(\mathbb{Z}_+)$ so f is not surjective. $\square$

Theorem: Let A be a set. There is no surjective function $g: A \to \mathscr{P}(A)$.

Proof: Let $g: A \to \mathscr{P}(A)$.

Define $S \in \mathscr{P}(A)$ as follows:

$$S = \{ a \mid a \in A - g(a) \} \subseteq A$$

i.e. S is the subset of A consisting of elements not contained in their image under g.

Now suppose $g(b) = S$ for some $b \in A$.

If $b \in S$, then $b \in A - g(b) = A - S$, a contradiction.

If $b \notin S$, then $b \notin A - g(b) = A - S$, so $b \in S$, again a contradiction.

Thus, $S \notin g(A)$, so g is not surjective. $\square$

Corollary: $\mathscr{P}(A)$ has different cardinality than A.

Corollary: If A is countably infinite, then $\mathcal{P}(A)$ is uncountable.

Ex: $\mathbb{R}$ is uncountable.

Pf: We'll show $(0,1)$ is uncountable, which will prove that $\mathbb{R}$ is uncountable, since $(0,1) \subseteq \mathbb{R}$.

Suppose $(0,1)$ is countably infinite. Then $\exists$ a bijection

$$f: \mathbb{Z}_+ \longrightarrow (0,1). \quad \text{Let } a_{ij} \text{ be the } j^{\text{th}} \text{ of } f(i).$$

[We can assume these are unique by not allowing decimals ending in infinitely repeating 9s. i.e. choose 0.1 over 0.0999....]

Then
$$\begin{aligned}
1 &\longmapsto 0.a_{11}a_{12}a_{13}\ldots \\
2 &\longmapsto 0.a_{21}a_{22}\ldots \\
3 &\longmapsto 0.a_{31}a_{32}\ldots
\end{aligned}$$

Define a number b w/ decimal expansion $b = 0.b_1b_2b_3\ldots$ as follows:

$$b_i = \begin{cases} 1 & \text{if } a_{ii} \neq 1 \\ 2 & \text{if } a_{ii} = 1. \end{cases}$$

Then $b_i \neq a_{ii}$ $\forall\, i \in \mathbb{Z}_+$, so $b \neq f(i)$ $\forall\, i \in \mathbb{Z}_+$.

Thus, f is not surjective, a contradiction. $\square$

Caution: We haven't shown that every real # has a unique decimal expansion, so this proof is more for fun and shouldn't be considered course material.